

Calculus III: Coursework 7
M.A.H. MacCallum 2007

The deadline is 4PM on Tuesday Nov 27th. Hand-in is to the green box on the ground floor of the Maths building.

1. Using Picard's method, find a solution in series of [20]

$$u' = x^3(u + u^2)$$

given that $u = 2$ at $x = 0$, up to terms of order x^{12} . Taking the x^{12} term as a good estimate of the error, estimate the range of x for which the the first three terms of this series give an answer accurate to 0.001 .

Answer We are given $u_0 = 2$. Using Picard,

$$\begin{aligned}u_1 &= 2 + \int_0^x x^3(2 + 2^2)dx = 2 + 3x^4/2 \\u_2 &= 2 + \int_0^x x^3(2 + \tfrac{3}{2}x^4 + (2 + \tfrac{3}{2}x^4)^2)dx \\&= 2 + \tfrac{3}{2}x^4 + \tfrac{15}{16}x^8 \dots \\u_3 &= 2 + \int_0^x x^3(2 + \tfrac{3}{2}x^4 + \tfrac{15}{16}x^8 + (2 + \tfrac{3}{2}x^4 + \tfrac{15}{16}x^8)^2)dx \\&= 2 + \tfrac{3}{2}x^4 + \tfrac{15}{16}x^8 + \tfrac{x^{12}}{12} \left(\tfrac{15}{16} + \tfrac{2 \cdot 15}{8} + \tfrac{36}{16} \right) \dots \\&= 2 + \tfrac{3}{2}x^4 + \tfrac{15}{16}x^8 + \tfrac{37}{64}x^{12} \dots\end{aligned}$$

To get the required accuracy, we need $\frac{37}{64}x^{12} < 0.001$ which gives approximately $x < 0.59$.

Marks: 4 for correct Picard method, 2, 4, 6 respectively for successive u_i (take 2 off if extra terms – which would be changed in the final answer – are retained), 4 for numerical part.

Comment: this equation can be solved exactly by separation of variables to give $u = 2e^{x^4/4}/(3 - 2e^{x^4/4})$ which of course has the same expansion.

Note: in the following questions on Frobenius series, take $x_0 = 0$ and say first what its nature is (ordinary, regular singular, ...). I do not expect you to give a formula for the n -th coefficient in the series, even if it can be done.

2. Using Frobenius's method, find the first 4 terms of each of the series solutions of [23]

$$2xu'' - 3u' + u = 0 .$$

Answer Here $x = 0$ is regular singular ($xf(x) = -3/2$, $x^2g = x/2$ in the standard form). Using formulae in the lectures the indicial equation is $c(c - 1) - 3c/2 = 0$, so

$c = 0$ or $c = 5/2$. Since the two values do not differ by an integer both series can be calculated without problems.

The recurrence relation (from formulae in lectures) is

$$[(r+c)(r+c-1) - 3(r+c)/2]a_r + a_{r-1}/2 = 0$$

which for $c = 0$ is

$$r(2r-5)a_r + a_{r-1} = 0$$

giving a series

$$a_0(1 + \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{18} \dots)$$

while for $c = 5/2$ we have

$$(2r+5)ra_r + a_{r-1} = 0$$

giving

$$a_0x^{5/2}(1 - \frac{x}{7} + \frac{x^2}{126} + \frac{x^3}{4158} + \dots)$$

Marks Regular singular point 3. Roots for c 4. Correct recurrence relation 4. 6 for each series (1 for the x term, 2 for x^2 correct, 3 for x^3 .)

3. Using Frobenius's method, find the series solutions of [17]

$$u'' - xu' + 2u = 0 ,$$

up to x^5 .

Answer $x = 0$ is an ordinary point, so $c = 0$ or $c = 1$. $xf = -x^2$ and $x^2g = 2x^2$ so the recurrence (for $c = 0$) is

$$r(r-1)a_r + [2 - (r-2)]a_{r-2} = 0 .$$

As expected a_1 is arbitrary. Moreover $a_4 = 0$. The general solution is

$$a_0(1 - x^2) + a_1x(1 - \frac{1}{6}x^2 - \frac{1}{120}x^4 \dots)$$

Marks: 3 for ordinary point, 4 for $c = 0$ or $c = 1$, 4 for correct recurrence relation, 3 each for the two series (1 for x^2 term, 2 for x^4 term correct – this being zero for the first series).

4. Applying Frobenius's method, solve the indicial equation for [16]

$$x^2u'' + x^2u' + (x-2)u = 0$$

and show that the series for the smaller root terminates. Give the resulting closed form solution.

Answer

Here $x = 0$ is a regular singular point. $xf = x$ and $x^2g = x - 2$ so the indicial equation is $c(c - 1) - 2 = 0$ giving $(c + 1)(c - 2) = 0$, so $c = -1$ or $c = 2$.

The recurrence relation is

$$[(r + c - 2)(r + c + 1)]a_r + a_{r-1}[(r + c - 1) + 1] = 0$$

so for $c = -1$, $r(r - 3)a_r + (r - 1)a_{r-1} = 0$. This obviously stops immediately (at $r = 1$) so the series for $c = -1$ gives just a_0/x . [In this case the other solution can also be given as an elementary function.]

Marks: 3 for regular singular. 4 for roots for c , 4 for correct general recurrence. 3 for any correct argument for termination and 2 for final solution.

5. Using Frobenius's method, find the index and recurrence relation for a series solution of [24]

$$xu'' + (1 - x)u' + ku = 0$$

(Laguerre's equation). Show that if k is a positive integer, the series becomes a polynomial. Give this polynomial for $k = 3$. [You should find this is one of the cases where the second solution involves a logarithm: you are not asked for this solution.]

Answer

Here $x = 0$ is again a regular singular point. It is easier to just use the form given and take

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n + c)(n + c - 1)a_n x^{(n+c-1)} + \sum_{n=0}^{\infty} (n + c)a_n x^{(n+c-1)} \\ &\quad - \sum_{n=0}^{\infty} (n + c)a_n x^{(n+c)} + k \left(\sum_{n=0}^{\infty} a_n x^{(n+c)} \right) \\ &= \sum_{n=-1}^{\infty} (n + c + 1)(n + c)a_{n+1} x^{(n+c)} + \sum_{n=-1}^{\infty} (n + c + 1)a_{n+1} x^{(n+c)} \\ &\quad - \sum_{n=0}^{\infty} (n + c)a_n x^{(n+c)} + \sum_{n=0}^{\infty} k a_n x^{(n+c)} \\ &= \sum_{n=-1}^{\infty} [(n + c + 1)^2 a_{n+1} - (n + c - k)a_n] x^{(n+c)} \end{aligned}$$

(though the standard formulae also work). The indicial equation (from taking $n = -1$) gives $c = 0$ twice, which from lectures means one solution involves $\ln x$. The other is given by the recurrence

$$(r + 1)^2 a_{r+1} - (r - k)a_r = 0.$$

If k is a positive integer the coefficient of a_r in the recurrence is 0 when $r = k$, so the solution is a polynomial of degree k . When $k = 3$ the recurrence gives

$$a_1 = -3a_0, \quad 4a_2 = -2a_1, \quad 9a_3 = -a_2, \quad a_4 = 0$$

so the polynomial is

$$a_0(1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3) .$$

Marks: 3 for regular singular. 4 for correct indicial equation and its roots, 2 for suitable remarks about second solution, 4 for general recurrent relation. Correct argument for polynomial 5. $k = 3$ total 6 (1,2,3 for the powers of x).